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PROBLEMS OF CONTROLLING BY THE RIGHT-HAND SIDES OF ELLIPTICAL SYSTEMS AND THEIR APPLICATION TO CONTROL OF THE STRESS-STRAIN STATE IN SHELLS*

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The problem of controlling by the right-hand sides of elliptical systems is considered for the case when the target function is a function of a maximum, and the set of admissible controls is not convex. The existence of a solution of an infinite-dimensional optimal control problem is proved and problems arising in the course of approximating the latter solution by solutions of the finite-dimensional problems are studied. The problems of controlling the stress-strain state of shells of revolution are used as examples. In this case the largest deviation of the stresses or displacements in the shell from the given values is minimized, with constraints imposed on the strength. The approach used here is related to /1/ where analogous problems were considered for the case of a quadratic target function and a convex set of admissible controls. Earlier, the problems of control by the right-hand sides were studied in /2-4/ for plates, and cylindrical axially loaded shells, and /5/ dealt with the problems of controlling by the stress-strain state of the plates in the minimax formulation, with the function of thickness used as the control.

1. Formulation of the problem. Existence of a solution. We shall begin with a physical description of the problem. Let a certain structure (plate or shell) be acted on, in addition to the fixed basic load f_0, by a system of loads f localised at a part of the surface and able to vary within specified limits. This system serves as the control. We require to find such a load f from amongst the admissible loads, that the maximum deviation of the displacements or other specified characteristics of the stress state of the structure from the given values is minimal. We consider problems of determining the minimum deflection of the structure, of the maximum unloading, etc. The load f is chosen so as to fall between the lower and upper limits, i.e. $f_- \leqslant f \leqslant f_+$, and such that the structure withstands the loading. The load f can simulate the action of certain special controls, as well as the reactions of the elastic supports and rigid edges. We can use the loads f to "tune" the structure during its use, to the action of varying loads f_0 on it. The problem under consideration corresponds to that of controlling the right-hand sides of the equations describing the state of the structure. We shall now give the abstract formulation of the problem.

Let X be a Hilbert space of functions over the field of real numbers, defined in a bounded domain $\Omega \subset R^n$. We shall consider the space X^* dual to X, as the space of controls. The state $u_f \in X$ of the control system is defined for the control $f \in X^*$ as a solution of the equation

$$a(u_f, v) = (f, v),\ \forall v \in X \tag{1.1}$$

Here $u, v \to a(u, v)$ is a bilinear form on $X \times X$, continuous and coercive, i.e.

$$a(u, u) \geqslant \alpha \| u \|_X^2, \alpha > 0,\ \forall u \in X \tag{1.2}$$

where (f, v) is the value of the functional $f \in X^*$ on the element $v \in X$. We note that in problems of structural mechanics the bilinear form a is generated by the potential energy of deformation, and f is the load acting on the structure. The function u_f solving the problem (1.1) determines the displacements of the structure.

According to the Lax-Milgram theorem /6/ the mapping $f \to u_f$ acting from X^* into X determined by the equation (1.1) is a homeomorphism. Let $\varphi : u \mapsto \varphi(u)$ denote a continuous mapping of X into $C(\Omega^c)$. Then the function

$$f \mapsto \varphi(u_f) \equiv \varphi_f \tag{1.3}$$

will represent a continuous mapping of X^* into $C(\Omega^c)$ as a combination of two continuous mappings. The target function has the form

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$$f \mapsto \psi(f) = \max_{x \in \Omega^c} \varphi_f(x) \tag{1.4}$$

where Ω^c is the closure of the region $\Omega \subset R^n$.

Taking into account the continuity of the mapping $f \mapsto \varphi_f(x)$ and the inequality

$$|\max_{x \in \Omega^c} F_1(x) - \max_{x \in \Omega^c} F_2(x)| \leqslant \max_{x \in \Omega^c} |F_1(x) - F_2(x)|, \quad \forall F_1, F_2 \in C(\Omega^c)$$

and remembering that a mapping continuous on a compact is uniformly continuous, we find that $f \mapsto \psi(f)$ maps X^* continuously into R. The problem of optimal control consists of finding q such that

$$q \in U_\partial, \ \psi(q) = \inf_{f \in U_\partial} \psi(f) \tag{1.5}$$

where $U_\partial \subset X^*$, U_∂ is the set of admissible controls.

Theorem 1. Let Y be a reflexive Banach space such that $Y \subset X^*$ (the inclusion $Y \to X^*$ is compact). If U_∂ is a bounded, sequentially weakly closed set in Y, then a solution of the problem (1.5) exists i.e. an element $q \in U_\partial$ can be found for which

$$\psi(q) = \inf_{f \in U_\partial} \psi(f)$$

Proof. Let $\{f_n\} \subset U_\partial$ be a minimizing sequence, i.e.

$$n \to \infty, \quad \psi(f_n) \to \inf_{f \in U_\partial} \psi(f) \tag{1.6}$$

From the boundedness of the set U_∂ and Y it follows that a subsequence $\{f_m\}$ can be separated from the sequence $\{f_n\}$ such, that $f_m \to f_0$ weakly in Y. Since the inclusion of Y in X^* is compact, then

$$f_m \to f_0 \text{ strongly in } X^* \tag{1.7}$$

Since U_∂ is sequentially weakly closed in Y, we have

$$f_0 \in U_\partial \tag{1.8}$$

From (1.7) we have $\lim \psi(f_m) = \psi(f_0)$, therefore by virtue of (1.6) we obtain

$$\psi(f_0) = \inf_{f \in U_\partial} \psi(f) \tag{1.9}$$

i.e. the function $q = f_0$ is a solution of the problem (1.5).

Note 1°. Since any closed convex set in a Banach space is sequentially weakly closed /6/, it follows that the theorem also holds in the case when U_∂ is a bounded convex set in Y.

2. Approximating the solution of the optimal control problem by solving a finite-dimensional problem. Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of finite-dimensional subspaces in Y satisfying the condition

$$\lim_{n \to \infty} \inf_{h \in H_n} \|h - \omega\|_Y = 0, \quad \forall \omega \in Y \tag{2.1}$$

The finite-dimensional problem consists of finding h_n such that

$$h_n \in H_n \cap U_\partial, \ \psi(h_n) = \inf_{f \in H_n \cap U_\partial} \psi(f) \tag{2.2}$$

It can easily be shown that the problem (2.2) reduces to the problem of mathematical programming. Indeed, let f_i $(i = 1, \ldots, n)$ be a basis in H_n. Substituting f in the target functional (1.4) by

$$f = \sum_{i=1}^{n} a_i f_i$$

we find that $f \to \psi(f)$ is a scalar function and $\mathbf{a} \mapsto J(\mathbf{a})$ where $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in R^n$, and the problem (2.2) reduces to that of finding a vector $\mathbf{b}$ such that

$$\mathbf{b} \in K_\partial \subset R^n, \ J(\mathbf{b}) = \inf_{\mathbf{a} \in K_\partial} J(\mathbf{a}) \tag{2.3}$$

$$K_\partial = \{\mathbf{a} \mid \mathbf{a} = (a_1, \ldots, a_n) \in R^n, \ f = \sum_{i=1}^{n} a_i f_i \in U_\partial\} \tag{2.4}$$

Let us denote by U_∂° the interior of the set U_∂ and fit U_∂ with a topology induced by the strong topology of the space Y.

Theorem 2. Let the conditions of Theorem 1 hold and $\{H_n\}_{n=1}^{\infty}$ be a sequence of finite-dimensional subspaces in Y satisfying the condition (2.1). Let also a sequence $\{g_n\}_{n=1}^{\infty}$ exist such that

$$g_n \in U_\partial^\circ, \forall n;\ g_n \to q \text{ strongly in } Y \tag{2.5}$$

where q is a solution of the problem (1.5). Then for sufficiently large n the problem (2.2) will have a solution h_n and

$$\lim \psi(h_n) = \psi(q) = \inf_{f \in U_\partial} \psi(f) \tag{2.6}$$

We can select from the sequence $\{h_n\}_{n=k}^{\infty}$ a subsequence $\{h_m\}_{m=1}^{\infty}$ such that $h_m \to q$ weakly in Y.

Proof. Taking into account (2.1) and (2.5) we find that at sufficiently large n (with $n \geqslant k$) the set $H_n \cap U_\partial$ is nonempty and a sequence $\{e_n\}_{n=k}^{\infty}$, exists such that

$$e_n \in H_n \cap U_\partial;\ \ e_n \to q \text{ strongly in } Y \tag{2.7}$$

From (2.7) it follows that

$$\lim \psi(e_n) = \psi(q) \tag{2.8}$$

The set $H_n \cap U_\partial$ is compact in Y and, since $f \to \psi(f)$ maps X^* continuously into R, it follows that the problem (2.2) has a solution at any $n \geqslant k$. Moreover, if the set $H_n \cap U_\partial$ is nonempty at any n, then the problem (2.2) has a solution at any n. From (1.5), (2.2) and (2.7) is follows that $\psi(e_n) \geqslant \psi(h_n) \geqslant \psi(q)$ and $\lim \psi(h_n) = \psi(q)$, i.e. (2.6) holds. The sequence $\{h_n\}_{n=k}^{\infty}$ is a minimizing sequence of the functional $f \to \psi(t), f \in U_\partial$. As in the proof of Theorem 1, we find that we can separate from the sequence $\{h_n\}_{n=k}^{\infty}$ a subsequence $\{h_m\}_{m=1}^{\infty}$ such that $h_m \to q$ weakly in Y.

Notes 1°. If the set U_∂ is convex, then the theorem also holds in the case when U_∂ contains at least one interior point x_0. Indeed, according to a known theorem /7/ every point of an open segment with end points x_0 and q is an interior point of the set U_∂.

2°. Consider the set

$$X_0 = \{q \mid q \in U_\partial,\quad \psi(q) = \inf_{f \in U_\partial} \psi(f)\}$$

Theorem 1 asserts that X_0 is nonempty. Theorem 2 gives a finite-dimensional approximation to one of the elements of the set X_0. Generally speaking, we can select from the sequence $\{h_n\}_{n=k}^{\infty}$ a subsequence $\{h_m\}_{m=1}^{\infty}$ converging weakly to various elements of X_0.

3°. The proof of the theorem indicates that the condition of existence of a sequence $\{g_n\}$ satisfying the conditions (2.5) can be replaced by the conditions (2.7). The latter may hold even when U_∂^0 is empty.

3. Controlling the stress-strain state of a shell of revolution. We consider as an example the problem of controlling the stress-strain state of a shell of revolution. The generalized solution $\omega' \in X$ satisfies the relation

$$a(\omega', \omega'') = \int_\Omega (f_0 + f)\,\omega'' AB\,d\varphi\,dz, \quad \forall \omega'' \in X \tag{3.1}$$

$$\begin{aligned} a(\omega', \omega'') = \frac{E_1}{1 - \nu_1\nu_2} \int_\Omega \Big\{ h \Big[&\varepsilon_{11}'\varepsilon_{11}'' + \nu_2(\varepsilon_{11}'\varepsilon_{22}'' + \varepsilon_{22}'\varepsilon_{11}'') + \\ &\frac{\nu_2}{\nu_1}\varepsilon_{22}'\varepsilon_{22}'' + \frac{1 - \nu_1\nu_2}{E_1} G\varepsilon_{12}'\varepsilon_{12}'' \Big] + \frac{h^3}{12}\Big[\gamma_{11}'\gamma_{11}'' + \nu_2(\gamma_{11}'\gamma_{22}'' + \\ &\gamma_{22}'\gamma_{11}'') + \frac{\nu_2}{\nu_1}\gamma_{22}'\gamma_{22}'' + 4\frac{1 - \nu_1\nu_2}{E_1} G\gamma_{12}'\gamma_{12}''\Big]\Big\} AB\,d\varphi\,dz \end{aligned}$$

Here (r, φ, z) are the cylindrical coordinates /8/, $\omega = (u, v, w)$ are the displacements of the point lying on the shell middle surface where the displacements are functions of φ and z, 2π-periodic in φ, $z \in [0, L]$ where L is the length of the shell, $\Omega = (0, 2\pi) \times (0, L)$, E_1, E_2 are the moduli of elasticity, ν_1 and ν_2 are the Poisson's ratios, G is the shear modulus, $h = h(\varphi, z)$ and $r = r(z)$ denote the thickness and the generatrix of the shell respectively, A and B are the coefficients of the first quadratic form, $\varepsilon_{ik}', \gamma_{ik}'$ and $\varepsilon_{ik}'', \gamma_{ik}''$ are the deformation tensor components generated by the displacements ω' and ω'' of the shell's middle surface and representing linear functions of ω', ω'' and their derivatives /9/, $X = V(\Omega)$ where $V(\Omega)$ is the closure in the norm

$$\|\omega\|_{V(\Omega)}^2 = \|u\|_{W_2^1(\Omega)}^2 + \|v\|_{W_2^1(\Omega)}^2 + \|w\|_{W_2^2(\Omega)}^2 \tag{3.2}$$

of the set of smooth functions in Ω, 2π-periodic in φ and satisfying the boundary conditions $J\omega = 0$, $J \in L(V_1(\Omega); Z)$ is the operator of the boundary conditions where Z is a Hilbert space. Here the operator J corresponds to the clamping of the shell in such a manner that it cannot be displaced as a rigid unit, i.e. for every $\omega \in V_1(\Omega)$ the conditions $a(\omega, \omega) = 0$ and $J\omega = 0$ imply that $\omega = 0$; $W_p^q(\Omega)$ are the Sobolev spaces and $L(Q; S)$ is a space of continuous linear mappings from the normed space Q into the normed space S.

In /8,10,11/ it was shown that when certain physically justified constraints are imposed on the quantities $\nu_1, \nu_2, E_1, E_2, G, r, h$, then the bilinear form a is symmetrical, continuous and coercive in $V(\Omega)$. Therefore every element $f \in (V(\Omega))^*$ has a unique corresponding function ω_f which solves the problem (3.1), and it is also assumed that f_0 is a fixed function belonging to $(V(\Omega))^*$.

Let the function (1.3) have the form

$$f \mapsto \varphi_f = F\omega_f^* \tag{3.3}$$

Here ω_f^* represents the averaging (contraction) of the displacements function ω_f which solves the problem (3.1), with a smooth kernel, the support of the latter of sufficiently small radius /12/. The function ω_f can be continued smoothly from Ω onto R^2, and $\omega_f^* \in [C^2(\Omega^c)]^3$.

The mapping F acting from $[C^2(\Omega^c)]^3$ to $C(\Omega^c)$ can be chosen in the form $F = F_1$ or $F = F_2$, where

$$F_1 : t = (t_1, t_2, t_3) \mapsto F_1 t = \sum_{i=1}^{3} \alpha_i |t_i - k_i| \tag{3.4}$$

$$F_2 : t = (t_1, t_2, t_3) \mapsto F_2 t = |I_2(t) - k|$$

Here α_i are nonnegative constants, $k_i = k_i(\varphi, z)$, $k = k(\varphi, z)$ are given functions belonging to $C(\Omega^c)$, $t \to I_2(t)$ is a continuous mapping from $[C^2(\Omega^c)]^3$ to $C(\Omega^c)$ corresponding to the second invariant of the stress tensor in the shell (at some surface parallel to the middle surface) and $t = (t_1, t_2, t_3)$ is the averaged displacement vector.

In the first case $(F = F_1)$ the target function defined by (1.4), (3.3) corresponds to the displacements closest, in the minimax sense, to the given displacements, and in the second case $(F = F_2)$ the target function corresponds to the characteristics of the stress state (second invariant of the stress tensor) closest, in the minimax sense, to the given characteristics. We assume that

$$Y = \{f \mid f \in [W_p^1(\Omega)]^3,\ p > 2,\ f|_{\Omega \setminus \Omega_0} = 0,\ \Omega_0 \subset \Omega\} \tag{3.5}$$

where Y is a Banach space with the norm of the space $[W_p^1(\Omega)]^3$. The set of admissible controls is chosen in the form

$$U_\partial = U^0 \tag{3.6}$$

where U^0 is the closure, in the norm of the space Y, of the set

$$U = \{f \mid f \in Y;\ \|f\|_Y < d;\ f_- < f < f_+ \ \text{in}\ \Omega^c;\ \Phi_s(f) < 0,\ s = 1, 2\} \tag{3.7}$$

Here f_- and f_+ are given elements of the space $[C(\Omega^c)]^3$:

$$\Phi_s(f) = \max_{(\varphi, z) \in \Omega^c} Q_{fs}(\varphi, z) \tag{3.8}$$

$$Q_{fs}(\varphi, z) = \sum_{ijkm} \alpha_{ijkm} \sigma_{ijs}^{(f_0+f)}(\varphi, z)\, \sigma_{kms}^{(f_0+f)}(\varphi, z) + \sum_{ij} \beta_{ij} \sigma_{ijs}^{(f_0+f)}(\varphi, z) - 1$$

$$\sigma_{ijs}^{(f_0+f)}(\varphi, z) = \sigma_{ijs}[\omega_{f_0+f}^*(\varphi, z)]$$

Here $\sigma_{ijs}^{(f_0+f)}$ are the stresses in the shell corresponding to the averaged displacement function $\omega_{f_0+f}^*$. When $s = 1$, the stresses are determined on the outer surface of the shell, and for $s = 2$ on the inner surface, and α_{ijkm}, β_{ij} are constants related to the material strength constants $i, j, k, m = 1, 2$. The functions $f \mapsto \Phi_s(f)$ $(s = 1, 2)$ characterize the state of stress of the shell corresponding to the load $f_0 + f$. When $\Phi_s(f) = 0$ $(s = 1, 2)$, the limit state of stress is reached, related to the strength criterion determined by the right-hand side of (3.8). We assume that f_-, f_+ and d in (3.7), as well as the mechanical characteristics of the material and shell dimensions are such, that the set U is nonempty. Using the inclusion theorems /12/ we find that conditions of Theorem 1 hold for the controls chosen in the form (3.5), (3.6), i.e. a solution of the problem (1.5) exists.

Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of the finite-dimensional subspaces in Y satisfying the condition of limit density (2.1). Every point of the set U is interior, therefore by virtue of (3.6) for every $f \in U_\partial$ there exists a sequence $\{g_n\}_{n=1}^{\infty}$ for which the conditions (2.5) hold. Thus the conditions of Theorem 2 are fulfilled.

We note that the regularization (averaging) of the solution ω_{f_0+f} of the problem (3.1) was carried out in order for the target function (1.4) to be meaningful. In accordance with the hypotheses of the mechanics of continuum and the known results concerning the convergence of the averaged functions, the regularizing operation leaves the solution of (3.1) practically unaffected, provided that the radius of averaging kernel is sufficiently small. In particular, in the case of $F = F_1$ with $\alpha_1 = \alpha_2 = 0$ the regularizing operation need not be used.

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